

A Lecture on Quantum Logic Gates

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Abstract

In this note we make a short review of constructions of n -repeated controlled unitary gates in quantum logic gates.

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1 Introduction

This is one of my lectures entitled “Introduction to Quantum Computation” given at Graduate School of Yokohama City University. The contents of lecture are based on the book [1] and review papers [2], [3]. The controlled NOT gate (more generally, controlled unitary gates) plays very important role in quantum logic gates to prove a universality. The constructions of controlled unitary gates or controlled-controlled unitary gates are clear and easy to understand. But the construction of general controlled unitary gates (n-repeated controlled unitary gates) seem, in my teaching experience, not easy to understand for young graduate students. I thought out some method to make the proof more accessible to them. I will introduce it in this note. Maybe it is, more or less, well-known in some field in Pure Mathematics, but we are too busy to study such a field leisurely. I believe that this note will make non-experts more accessible to quantum logic gates.

2 Some Identity on Z_2

Let us start with the mod 2 operation in Z_2 : for $x, y \in Z_2$

$$x \oplus y = x + y \pmod{2}. \quad (1)$$

From the relations

$$0 \oplus 0 = 0, \quad 0 \oplus 1 = 1, \quad 1 \oplus 0 = 1, \quad 1 \oplus 1 = 0,$$

it is easy to see

$$x \oplus y = x + y - 2xy, \text{ or } x + y - x \oplus y = 2xy. \quad (2)$$

We note that $x \oplus 0 = x$, $x \oplus 1 = 1 - x$, $x \oplus x = 2x - 2x^2 = 2x(1 - x) = 0$.

From $x + y - x \oplus y = 2xy$ we have

$$x + y + z - (x \oplus y + x \oplus z + y \oplus z) + x \oplus y \oplus z = 4xyz \quad (3)$$

for $x, y, z \in Z_2$. The proof is easy, so we leave it to the readers. Moreover we have

$$\begin{aligned} & x + y + z + w - (x \oplus y + x \oplus z + x \oplus w + y \oplus z + y \oplus w + z \oplus w) + (x \oplus y \oplus z + \\ & x \oplus y \oplus w + x \oplus z \oplus w + y \oplus z \oplus w) - x \oplus y \oplus z \oplus w = 8xyzw \end{aligned} \quad (4)$$

for $x, y, z, w \in Z_2$. But is this proof so easy ?

Now we define a function

$$\begin{aligned} F_n(x_1, \dots, x_n) &= \sum_{i=1}^n x_i - \sum_{i<j}^n x_i \oplus x_j + \sum_{i<j<k}^n x_i \oplus x_j \oplus x_k - \dots \\ &+ (-1)^{n-2} \sum_{i=1}^n x_1 \oplus \dots \oplus \check{x}_i \oplus \dots \oplus x_n + (-1)^{n-1} x_1 \oplus \dots \oplus x_n \end{aligned} \quad (5)$$

for $x_1, \dots, x_n \in Z_2$. From (2), (3) and (4) we have $F_2(x_1, x_2) = 2x_1x_2$ and $F_3(x_1, x_2, x_3) = 4x_1x_2x_3$ and $F_4(x_1, x_2, x_3, x_4) = 8x_1x_2x_3x_4$. From these relations it is easy to conjecture

Proposition A

$$F_n(x_1, x_2, \dots, x_n) = 2^{n-1}x_1x_2 \dots x_n. \quad (6)$$

This is well-known [4], but I don't know the usual proof (in [4] there is no proof). This proof may be not easy for non-experts against the claim in the book [1], see pp. 30-31. Here let us introduce a new (?) method to prove this. For that we must make some mathematical preparations. First let us extend the operation \oplus in Z_2 to an operation $\tilde{\oplus}$ in Z : for $x, y \in Z$

$$x \tilde{\oplus} y \equiv x + y - 2xy. \quad (7)$$

Of course $x \tilde{\oplus} y = x \oplus y$ for $x, y \in Z_2$. Here we list some important properties of this operation :

Lemma 1 For $x, y, z \in Z$

$$\begin{aligned} x \tilde{\oplus} y &= y \tilde{\oplus} x, \\ (x \tilde{\oplus} y) \tilde{\oplus} z &= x \tilde{\oplus} (y \tilde{\oplus} z), \\ x \tilde{\oplus} z + y \tilde{\oplus} z &= (x + y) \tilde{\oplus} z + z, \\ x \tilde{\oplus} z - y \tilde{\oplus} z &= (x - y) \tilde{\oplus} z - z \end{aligned} \quad (8)$$

We also note that $x \tilde{\oplus} 0 = x$, $x \tilde{\oplus} 1 = 1 - x$, $x \tilde{\oplus} x = 2x(1 - x)$.

What we want to prove in this section is the following recurrent relation

Proposition B For $x_1, \dots, x_n, x_{n+1} \in Z_2$

$$F_{n+1}(x_1, \dots, x_n, x_{n+1}) = F_n(x_1, \dots, x_n) + x_{n+1} - F_n(x_1, \dots, x_n) \tilde{\oplus} x_{n+1}. \quad (9)$$

If we can prove this, then it is easy to see from the definition of $\tilde{\oplus}$

$$F_{n+1}(x_1, \dots, x_n, x_{n+1}) = 2x_{n+1}F_n(x_1, \dots, x_n). \quad (10)$$

From this we have Proposition A. Before giving the proof to Proposition B we make some preliminaries.

Lemma 2 For $x_1, \dots, x_n, z \in Z_2$

$$\sum_{i=1}^n x_i \oplus z = \left(\sum_{i=1}^n x_i \right) \tilde{\oplus} z + (n-1)z, \quad (11)$$

$$\sum_{i=1}^n (-1)^{i-1} x_i \oplus z = \left(\sum_{i=1}^n (-1)^{i-1} x_i \right) \tilde{\oplus} z - \frac{1 + (-1)^n}{2} z. \quad (12)$$

The proof is straightforward from Lemma 1.

Lemma 3 For $n \geq 2$

$$\sum_{i=1}^{n-1} (-1)^i ({}_nC_i - 1) = -\frac{1 + (-1)^n}{2}. \quad (13)$$

The proof is as follows :

$$\begin{aligned} \text{Left hand side} &= \sum_{i=1}^{n-1} (-1)^i {}_nC_i + \sum_{i=1}^{n-1} (-1)^{i+1} \\ &= \sum_{i=0}^n (-1)^i {}_nC_i - \{1 + (-1)^n\} + \sum_{i=0}^{n-2} (-1)^i \\ &= (1-1)^n - \{1 + (-1)^n\} + \frac{1 - (-1)^{n-1}}{2} \\ &= -\{1 + (-1)^n\} + \frac{1 + (-1)^n}{2} \\ &= -\frac{1 + (-1)^n}{2}. \quad \heartsuit \end{aligned}$$

First of all let us show my idea to prove Proposition B with a simple example.

$$F_3(x_1, x_2, x_3) = x_1 + x_2 + x_3 - (x_1 \oplus x_2 + x_1 \oplus x_3 + x_2 \oplus x_3) + x_1 \oplus x_2 \oplus x_3$$

$$\begin{aligned}
&= x_1 + x_2 - x_1 \oplus x_2 + x_3 - \{x_1 \oplus x_3 + x_2 \oplus x_3 - x_1 \oplus x_2 \oplus x_3\} \\
&= F_2(x_1, x_2) + x_3 - \{(x_1 + x_2) \tilde{\oplus} x_3 + x_3 - x_1 \oplus x_2 \oplus x_3\} \\
&= F_2(x_1, x_2) + x_3 - \{(x_1 + x_2 - x_1 \oplus x_2) \tilde{\oplus} x_3 - x_3 + x_3\} \\
&= F_2(x_1, x_2) + x_3 - F_2(x_1, x_2) \tilde{\oplus} x_3. \quad \heartsuit
\end{aligned}$$

Let us start the proof of Proposition B.

$$\begin{aligned}
F_{n+1}(x_1, \dots, x_n, x_{n+1}) &= \sum_{i=1}^{n+1} x_i - \sum_{i < j}^{n+1} x_i \oplus x_j + \sum_{i < j < k}^{n+1} x_i \oplus x_j \oplus x_k - \dots \\
&\quad + (-1)^{n-1} \sum_{i=1}^{n+1} x_1 \oplus \dots \oplus \check{x}_i \oplus \dots \oplus x_{n+1} + (-1)^n x_1 \oplus \dots \oplus x_n \oplus x_{n+1} \\
&= F_n(x_1, \dots, x_n) + x_{n+1} - \sum_{i=1}^n x_i \oplus x_{n+1} + \sum_{i < j}^n x_i \oplus x_j \oplus x_{n+1} - \dots \\
&\quad + (-1)^{n-1} \sum_{i=1}^n x_1 \oplus \dots \oplus \check{x}_i \oplus \dots \oplus x_n \oplus x_{n+1} + (-1)^n x_1 \oplus \dots \oplus x_n \oplus x_{n+1} \\
&= F_n(x_1, \dots, x_n) + x_{n+1} \\
&\quad - \left\{ \left(\sum_{i=1}^n x_i \right) \tilde{\oplus} x_{n+1} + ({}_nC_1 - 1)x_{n+1} \right\} \\
&\quad + \left\{ \left(\sum_{i < j}^n x_i \oplus x_j \right) \tilde{\oplus} x_{n+1} + ({}_nC_2 - 1)x_{n+1} \right\} \\
&\quad \dots \\
&\quad + (-1)^{n-1} \left\{ \left(\sum_{i=1}^n x_1 \oplus \dots \oplus \check{x}_i \oplus \dots \oplus x_n \right) \tilde{\oplus} x_{n+1} + ({}_nC_{n-1} - 1)x_{n+1} \right\} \\
&\quad + (-1)^n x_1 \oplus \dots \oplus x_n \oplus x_{n+1} \\
&= F_n(x_1, \dots, x_n) + x_{n+1} - \left(\sum_{i=1}^n x_i \right) \tilde{\oplus} x_{n+1} + \left(\sum_{i < j}^n x_i \oplus x_j \right) \tilde{\oplus} x_{n+1} - \dots \\
&\quad + (-1)^{n-1} \left(\sum_{i=1}^n x_1 \oplus \dots \oplus \check{x}_i \oplus \dots \oplus x_n \right) \tilde{\oplus} x_{n+1} + (-1)^n x_1 \oplus \dots \oplus x_n \oplus x_{n+1} \\
&\quad + \left\{ \sum_{i=1}^{n-1} (-1)^i ({}_nC_i - 1) \right\} x_{n+1} \\
&= F_n(x_1, \dots, x_n) + x_{n+1} - \left\{ \sum_{i=1}^n x_i - \sum_{i < j}^n x_i \oplus x_j + \dots \right. \\
&\quad \left. + (-1)^{n-2} \sum_{i=1}^n x_1 \oplus \dots \oplus \check{x}_i \oplus \dots \oplus x_n + (-1)^{n-1} x_1 \oplus \dots \oplus x_n \right\} \tilde{\oplus} x_{n+1} \\
&\quad + \frac{1 + (-1)^n}{2} x_{n+1} - \frac{1 + (-1)^n}{2} x_{n+1} \quad \text{by Lemma 2 and Lemma 3}
\end{aligned}$$

$$= F_n(x_1, \dots, x_n) + x_{n+1} - F_n(x_1, \dots, x_n) \tilde{\oplus} x_{n+1}. \quad \heartsuit$$

One word : I introduced one method to prove Proposition A. Of course we have an another one [5], but in my teaching experience my method was popular among young graduate students.

3 General Controlled Unitary Gates

Let a basis of 1-qubit space \mathcal{C}^2 be $\{|0\rangle, |1\rangle\}$.,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and 2-qubit space $\mathcal{C}^2 \otimes \mathcal{C}^2$ be

$$\mathcal{C}^2 \otimes \mathcal{C}^2 = \text{Vect}_C\{|0, 0\rangle, |0, 1\rangle, |1, 0\rangle, |1, 1\rangle\} \cong \mathcal{C}^4$$

where $|i, j\rangle \equiv |i\rangle \otimes |j\rangle$ for $i, j = 0, 1$.

The controlled NOT operation is defined as

$$\begin{aligned} \text{C-NOT :} \quad & |0, 0\rangle \rightarrow |0, 0\rangle, \quad |0, 1\rangle \rightarrow |0, 1\rangle, \\ & |1, 0\rangle \rightarrow |1, 1\rangle, \quad |1, 1\rangle \rightarrow |1, 0\rangle \end{aligned} \tag{14}$$

and , therefore, the matrix representation is

$$\text{C-NOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{15}$$

and represented graphically as

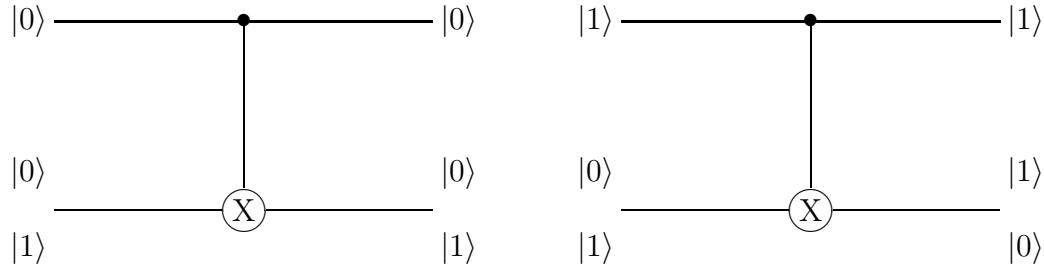


Figure 1

Let U be an arbitrarily unitary matrix in $U(2)$. Then the controlled unitary gates are defined as

$$\begin{aligned} \text{C-U : } \quad & |0, 0\rangle \rightarrow |0, 0\rangle, \quad |0, 1\rangle \rightarrow |0, 1\rangle, \\ & |1\rangle \otimes |0\rangle \rightarrow |1\rangle \otimes (U|0\rangle), \quad |1\rangle \otimes |1\rangle \rightarrow |1\rangle \otimes (U|1\rangle) \end{aligned} \quad (16)$$

more briefly,

$$\text{C-U : } |x\rangle \otimes |y\rangle \rightarrow |x\rangle \otimes (U^x|y\rangle) \quad \text{for } x, y \in Z_2 \quad (17)$$

and represented graphically as

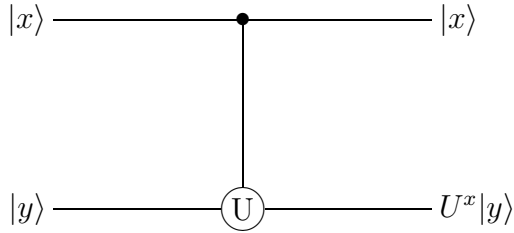


Figure 2

If $U = \sigma_2 = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then the controlled unitary gate is just controlled NOT gate.

The controlled-controlled unitary gates are defined as

$$\text{C-C-U : } |x\rangle \otimes |y\rangle \otimes |z\rangle \rightarrow |x\rangle \otimes |y\rangle \otimes (U^{xy}|z\rangle) \quad \text{for } x, y, z \in Z_2. \quad (18)$$

The controlled-controlled unitary gates are constructed by making use of several controlled unitary gates and controlled NOT gates : Let U be an arbitrarily unitary matrix in $U(2)$ and V a unitary one in $U(2)$ satisfying $V^2 = U$. Then by relation (2)

$$V^{x+y-x\oplus y} = V^{2xy} = (V^2)^{xy} = U^{xy}, \quad (19)$$

controlled-controlled U gate is graphically represented as

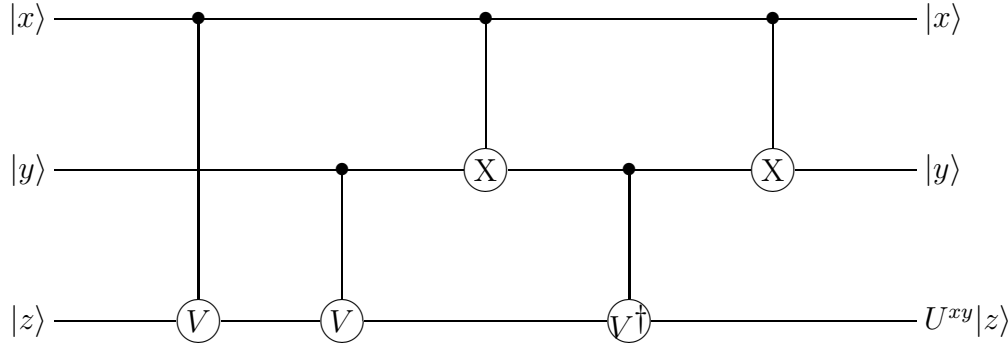


Figure 3

The controlled-controlled-controlled unitary gates are constructed by the following : Let U be an arbitrarily unitary matrix in $U(2)$ and V be a unitary one in $U(2)$ satisfying $V^4 = U$. Then making use of (3)

$$V^{x+y+z-(x\oplus y+x\oplus z+y\oplus z)+x\oplus y\oplus z} = V^{4xyx} = U^{xyz}, \quad (20)$$

controlled-controlled-controlled U gate is graphically represented as

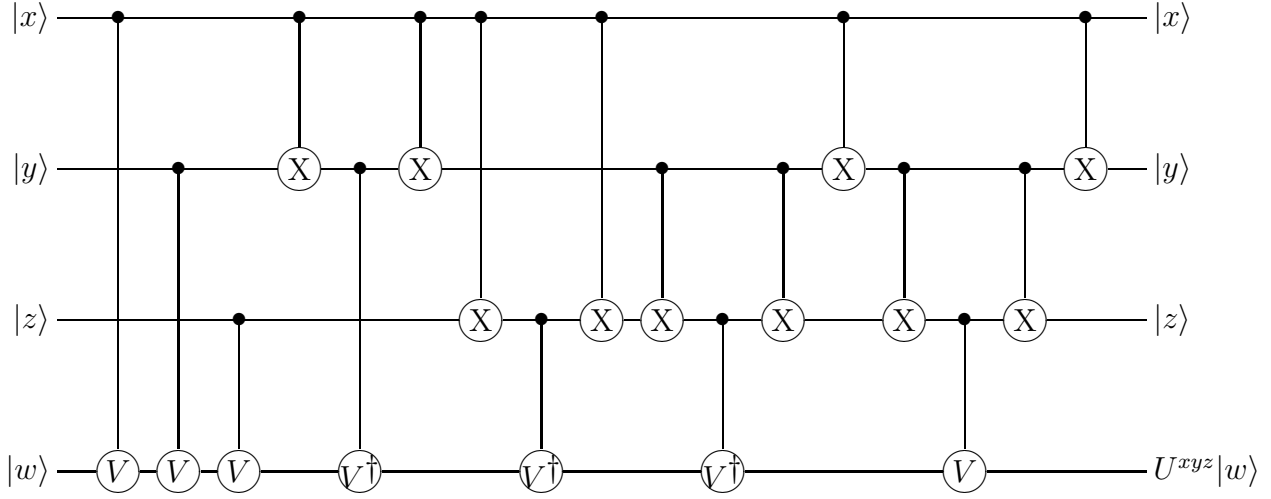


Figure 4

The general controlled unitary gates are constructed by the following : Let U be an arbitrarily unitary matrix in $U(2)$ and V be a unitary one in $U(2)$ satisfying $V^{2^{n-1}} = U$. Then making use of relation (6)

$$V^{F_n(x_1, x_2, \dots, x_n)} = V^{2^{n-1} x_1 x_2 \dots x_n} = U^{x_1 x_2 \dots x_n}, \quad (21)$$

the construction of n -repeated controlled U gate is as follows : For example the block implementing $V^{x_i \oplus x_j \oplus x_k}$ is graphically constructed as

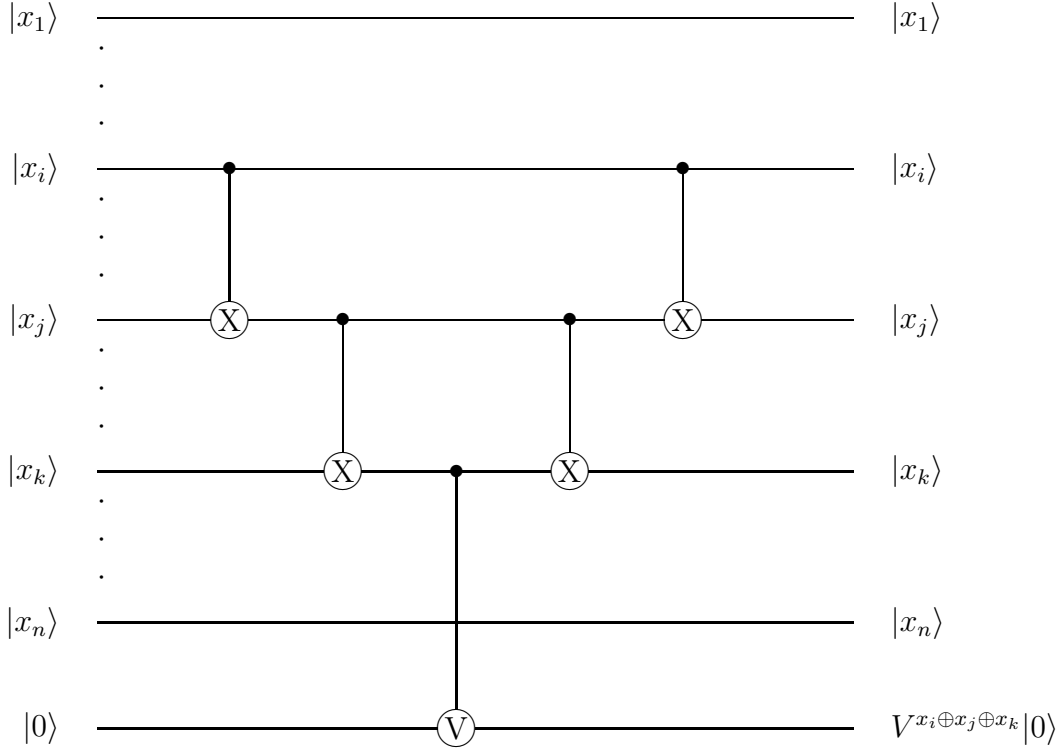


Figure 5

By combining these blocks like Figure 3 and Figure 4 we have the n -repeated controlled unitary gates. But as emphasized in [4] this construction is not efficient.

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